International Journal of Theoretical Physics, Vol. 34, No. 8, 1995

Unsharp Quantum Logics

Maria Luisa Dalla Chiara[1]

Received October 5, 1994

The event-structure of a state–event system, containing unsharp elements, can be described either as a *regular involutive bounded poset*, or alternatively as an *unsharp orthoalgebra* (called also *difference poset* or *effect algebra*). Such structures give rise to different forms of *unsharp quantum logics*.

A basic aim of the unsharp approaches to quantum theory (QT) (Busch *et al.*, 1991; Cattaneo and Laudisa, 1994; Davies, 1983) is to provide a mathematization of some ambiguous, nebulous aspects which seem to be characteristic of "concrete reality." All this enables us to fill a gap between an *exact* mathematics and a *fuzzy* experimental world.

As is well known, according to the standard logicoalgebraic approaches, any physical theory $\mathbb{T}$ is associated to a collection of *state–event systems* $\langle S, \mathscr{E}\rangle$, where S contains the *states* that a physical system described by the theory may assume and $\mathscr{E}$ contain the *events* that may occur in the system.

Let $s, t, \ldots$ represent elements of S; while $E, F, \ldots$ are elements of $\mathscr{E}$. The minimal abstract conditions that it seems reasonable to require for any quantum mechanical $\langle S, \mathscr{E}\rangle$ are the following:

1. $\forall s \forall E$: $s(E) \in [0, 1]$
   (any state associates a generalized probability value to any event).
2. *Weak extensionality*:
   $\forall E, F$: $\forall s[s(E) = s(F)] \Rightarrow E = F$
   $\forall s, t$: $\forall E[s(E) = t(E)] \Rightarrow s = t$
   (events that are probabilistically indiscernible are identified, and similarly for states).

[1]Dipartimento di Filosofia, Università di Firenze, Florence, Italy. E-mail: dachiara@mailserver.idg.fi.cnr.it.

3. $\mathscr{E}$ is closed under a weak complement operation $'$ (called also *supplement*) such that:
   $\forall E \forall s$: $s(E') = 1 - s(E)$
4. $\mathscr{E}$ contains a *certain event* $\mathbf{1}$ such that:
   $\forall s$: $s(\mathbf{1}) = 1$
   Let $\mathbf{0} := \mathbf{1}'$ be the *impossible event.*

This enables us to define a partial order relation $\sqsubseteq$ and an orthogonality relation $\perp$ on $\mathscr{E}$ as follows.

Definition 1. $E \sqsubseteq F \Leftrightarrow \forall s[s(E) \leq S(F)]$.

Definition 2. $E \perp F \Leftrightarrow E \sqsubseteq F'$.

As a consequence one obtains that the structure $\langle \mathscr{E}, \sqsubseteq, ', \mathbf{1}, \mathbf{0} \rangle$ is an *involutive bounded regular poset*. In other words:

(a) $\sqsubseteq$ is a partial order with maximum $\mathbf{1}$ and minimum $\mathbf{0}$.
(b) $'$ is an involution: $(E = E''$; $E \sqsubseteq F \Rightarrow F' \sqsubseteq E')$.
(c) $E \perp E$ and $F \perp F \Rightarrow E \perp F$ (*regularity*).

One can also require that a partial sum $\boxplus$ is defined on $\mathscr{E}$:

5. $E \perp F \Leftrightarrow E \boxplus F \in \mathscr{E}$ and $\forall s[s(E \boxplus F) = s(E) + s(F)]$.

As a consequence one obtains that the structure $\langle \mathscr{E}, \boxplus, \mathbf{1}, \mathbf{0} \rangle$ is an *unsharp orthoalgebra*. In other words, $\boxplus$ is a partial binary operation satisfying the following conditions [where $\exists(E \boxplus F)$ means that $\boxplus$ is defined for E, F]:

(a) *Weak commutativity:*
$\exists(E \boxplus F) \Rightarrow \exists(F \boxplus E)$ and $E \boxplus F = F \boxplus E$
(b) *Weak associativity:*
$[\exists(F \boxplus G)$ and $\exists(E \boxplus (F \boxplus G))] \Rightarrow [\exists(E \boxplus F)$ and $\exists((E \boxplus F) \boxplus G))$ and $E \boxplus (F \boxplus G) = (E \boxplus F) \boxplus G]$
(c) *Strong excluded middle:*
For any E, there exists a unique F s.t. $E \boxplus F = \mathbf{1}$
(d) *Weak consistency:*
$\exists(E \boxplus \mathbf{1}) \Rightarrow E = \mathbf{0}$

Unsharp orthoalgebras have been also called *effect algebras* (Foulis and Bennett, n.d.) or *weak orthoalgebras* (Giuntini and Greuling, 1989). One can prove that the concept of unsharp orthoalgebra is equivalent to the notion of *difference poset* (or D-poset), investigated by Kôpka and Chovanec (1994), Dvurečenskij (1994), and Dvurečenskij and Pulmannová (1994).

A *sharp orthoalgebra* (or simply an *orthoalgebra*) satisfies besides conditions (a)–(c), the strong *noncontradiction principle*:

(d′) $\exists(E \boxplus E) \Rightarrow E = \mathbf{0}$.

As is well known, canonical Hilbert-space exemplifications of state–event systems $\langle S, \mathcal{E}\rangle$ can be obtained by taking as S the set of the density operators in the Hilbert space $\mathcal{H}$ (associated to the physical system under investigation), whereas $\mathcal{E}$ can be identified either with the set $\mathcal{P}(\mathcal{H})$ of the projections of $\mathcal{H}$ or, alternatively, with the set $\mathcal{E}(\mathcal{H})$ of the *effects* of $\mathcal{H}$.

The set of all effects $\mathcal{E}(\mathcal{H})$ can be naturally structured as an unsharp orthoalgebra $\langle \mathcal{E}(\mathcal{H}), \boxplus, \mathbf{1}, \mathbf{0}\rangle$, where:

- $\exists(E \boxplus F) \Leftrightarrow E + F \in \mathcal{E}(\mathcal{H})$.
- $\exists(E \boxplus F) \Rightarrow E \boxplus F = E + F$.
- $\mathbf{0} = \mathbb{O}$ (the null projection).
- $\mathbf{1} = \mathbb{1}$ (the identity projection).

This structure is not a sharp orthoalgebra, since the strong contradiction principle is violated. For instance: the semitransparent operator $\frac{1}{2}\mathbb{1}$ is an effect (to which any state assigns probability 1/2); further, $\exists(\frac{1}{2}\mathbb{1} \boxplus \frac{1}{2}\mathbb{1})$ and $\frac{1}{2}\mathbb{1} \neq \mathbb{O}$.

In contrast to $\mathcal{E}(\mathcal{H})$, $\mathcal{P}(\mathcal{H})$ gives rise to a sharp orthoalgebra. Hence, effects may be generally regarded as a kind of unsharp generalization of projections.

How can we define, in an abstract way, the distinction between sharp and unsharp events? Let us first notice that from the semantic point of view, events are naturally thought of as *intensions*. What is the *extension* of an event E? Different ideas of extension can be proposed.

Definition 3 (The *r-extension* of E [$Ext^r(E)$]). Let $r \in [0, 1]$. Then

$$Ext^r(E) = \{s \in S \,|\, s(E) = r\}$$

In other words, $Ext^r(E)$ is the set of states that assign probability r to E.

Let us call $Ext^1(E)$ the *positive extension* (or the *positive certainty domain*) of E. Similarly, $Ext^0(E)$ will represent the *negative extension* (or *negative certainty domain*) of E.

Definition 4. The *simple extension* (or *simple proposition*) of E is $Ext^1(E)$. In other words, the simple extension of E is the set of all states in S which certainly verify E.

This fairly corresponds to the notion of *proposition* that is generally adopted in the usual possible-world semantics, where the *extensional meaning* of a sentence is identified with the set of *possible worlds* which verify our sentence. Simple extensions (or propositions) correspond to a somewhat rough idea of extension that completely neglects what happens for all probability values different from the certain value. A finer definition can be obtained as follows:

Definition 5. The *generalized* (or *infinite*) *extension* is a function Ext_∞ s.t. Ext_∞: $\mathscr{E} \to \mathfrak{P}(S)^{[0,1]}$ and

$$Ext_\infty(E)(r) = Ext^r(E)$$

for any $E \in \mathscr{E}$ and $r \in [0, 1]$. Here $\mathfrak{P}(S)$ is the power set of S.

Needless to say, also simple extensions may be trivially described as functions Ext_1 s.t. Ext_1: $\mathscr{E} \to \mathfrak{P}(S)^{\{1\}}$. It might be useful to consider also an intermediate notion between simple and infinite extensions. [This has been applied in the semantic characterization for a form of Brouwer–Zadeh logic (Dalla Chiara *et al.*, 1993).]

Definition 6. The *yes–no extension* is a function Ext_2 s.t. Ext_2: $\mathscr{E} \to \mathfrak{P}(S)^{\{0,1\}}$ and

$$Ext_2(E)(1) = Ext^1(E)$$

$$Ext_2(E)(0) = Ext^0(E)$$

In other words, the yes–no extension of E associates to 1 the positive extension of E, and to 0 the negative one. As a consequence, we obtain that simple extensions $[Ext^1(E)]$ are represented by sets of states, yes–no extensions are represented by pairs of sets of states $[\langle Ext^1(E), Ext^0(E)\rangle]$, and generalized extensions correspond to infinite classes of sets of states $(\{Ext^r(E)\}_{r\in[0,1]})$.

Let $\overline{\mathscr{E}}^\infty$, $\overline{\mathscr{E}}^2$, $\overline{\mathscr{E}}^1$ be, respectively, the class of all generalized, yes–no, simple extensions of $\mathscr{E}$. And let $\overline{\mathscr{E}}^i$ (where $i \in \{1, 2, \infty\}$) represent any of our three extension-sets.

Generally, events are not determined by their simple extensions: in other words, different events may have one and the same simple extension. In the same way, different concepts (say "equiangle triangle," "equilateral triangle") may have the same extension in the usual semantics. At the same time, events are always determined by their generalized extensions; for, we have assumed a weak extensionality principle. In some very peculiar situations, it may happen that events are determined by their simple extensions. In other words:

$$\forall E \in \mathscr{E}: \quad E = F \Leftrightarrow Ext^1(E) = Ext^1(F).$$

Two important examples of this kind are the following:

1. *Classical probability theory.* Take $\mathscr{E}$ as any measurable σ-field of sets. Take S as the class of all probability measures on $\mathscr{E}$.
2. *Standard quantum mechanics.* Take $\mathscr{E}$ as the class $\mathcal{P}(\mathcal{H})$ of all orthogonal projections on the Hilbert space $\mathcal{H}$. Take S as the set of all density matrices in $\mathcal{H}$.

In such cases, Ext^1 induces a structure on the extension set $\overline{\mathscr{E}^1}$ ($\langle \overline{\mathscr{E}^1}, \subseteq, ', \bar{\mathbf{1}}, \bar{\mathbf{0}} \rangle$) which is trivially isomorphic to the original event-structures. Of course, the partial order relation collapses, here, into the set-theoretic inclusion; $'$ becomes a set-theoretic operation, generally different from the complementation. Should $\overline{\mathscr{E}^1}$ be closed under the set-theoretic intersections, one immediately obtains a lattice structure for $\overline{\mathscr{E}^1}$, which will be automatically transferred to $\mathscr{E}$ by isomorphism. This is the well-known situation which arises in standard QM, where quantum propositions are indifferently identified either with projections (events) or with closed subspaces (corresponding to simple propositions). Just this correspondence has been described by Foulis and Randall as one of the "metaphysical disasters" of orthodox Hilbert space quantum theory. The extensional collapse breaks down if we decide to enlarge the set of our quantum events and to identify $\mathscr{E}$ with $\mathscr{E}(\mathscr{H})$ (the set of all effects of $\mathscr{H}$). Effects are determined neither by their simple nor by their yes–no extensions.

Let us now propose three possible abstract definitions for the notion of sharpness.

Let $\langle \mathscr{E}, S \rangle$ be a state–event system.

Definition 7. An event E is *weakly sharp* iff E satisfies the noncontradiction principle. In other words, $E \sqcap E' = \mathbf{0}$ if $E \sqcap E'$ (the *inf* between E and E') exists in $\mathscr{E}$.

Definition 8. E is *semistrongly sharp* iff either $E = \mathbf{0}$ or for at least one state s: $s(E) = 1$.

In other words, if not impossible, our event is certainly satisfied by at least one state.

Definition 9. E is *strongly sharp* iff the negative extension of E is the maximal one.

In other words,

$$s \in Ext^0(E) \Leftrightarrow s \perp Ext^1(E)$$

where $s \perp X$ means $\forall t \in X\, \exists F \in \mathscr{E}[s(F) = 1 \text{ and } t(F) = 0]$.

One can easily check that

$$\text{strong sharpness} \Rightarrow \text{semistrong sharpness} \Rightarrow \text{weak sharpness}$$

but not the other way around.

Proper effects in QM may violate all three conditions. As an example, let us think of the *semitransparent effect* $\frac{1}{2}\mathbb{1}$.

Both the event and the proposition structures give rise to particular models of quantum logic (QL). Generally, the intensional and the corresponding extensional structures are not associated to one and the same logic (think of the case of the effect structures and the Brouwer–Zadeh logics!). As is well known, the standard projection structure is a model for orthodox QL (Birkhoff–von Neumann), which is a sharp logic, where no violation of the noncontradiction principle is admitted. At the same time, orthodox QL is also a total logic, since its basic logical constants behave as total operations (universally defined). Event structures which may contain also unsharp elements (like the effect structures in Hilbert space QT) have suggested different forms of paraconsistent and fuzzy QL where the noncontradiction principle admits violations. It may also happen that the basic logical constants are not universally defined. For instance, the conjunction of two meaningful sentences does not generally have a well-determined meaning. This gives rise to a *partial unsharp quantum logic* (Dalla Chiara and Giuntini, 1994, n.d.). The study of the correlations among these different unsharp logics is an object of research that is still in progress (Dalla Chiara and Giuntini, n.d.; Giuntini, n.d.).

REFERENCES

Busch, P., Lahti, P., and Mittelstaedt, P. (1991). *The Quantum Theory of Measurement*, Springer-Verlag, Berlin.

Cattaneo, G., and Laudisa, F. (1994). Axiomatic unsharp quantum mechanics, *Foundations of Physics*, **24**, 631–683.

Dalla Chiara, M. L., and Giuntini, R. (1994). Partial and unsharp quantum logics, *Foundations of Physics*, **24**, 1161–1177.

Dalla Chiara, M. L., and Giuntini, R. (n.d.). The logics of orthoalgebras, *Studia Logica*, to appear.

Dalla Chiara, M. L., Cattaneo, G., and Giuntini, R. (1993). Fuzzy-intuitionistic quantum logic, *Studia Logica*, **52**, 1–24.

Davies, E. B. (1983). *Quantum Theory of Open Systems*, Academic Press, New York.

Dvurečenskij, A. (1994). Tensor product of difference posets, *Proceedings of the American Mathematical Society*, to appear.

Dvurečenskij, A., and Pulmannová, S. (1994). Difference posets, effects, and quantum measurements, *International Journal of Theoretical Physics*, **33**, to appear.

Foulis, D. J., and Bennett, M. K. (n.d.). Effect algebras and unsharp quantum logics, *Foundations of Physics*, to appear.

Giuntini, R. (n.d.). Quantum MV algebras, preprint.

Giuntini, R., and Greuling, H. (1989). Toward a formal language for unsharp properties, *Foundations of Physics*, **20**, 931–935.

Kôpka, F., and Chovanec, F. (1994). D-posets, *Mathematica Slovaca*, **44**, 21–34.